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Construction of the exact solution of the ring-plate problem by solving functional equations

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Abstract. This paper presents a method of constructing the exact solution of the ring-plate problem. The method is based on the general solution formula (nonseries form) of the biharmonic equation. The method changes solving the boundary-value problems of the ring plate into solving three functional equations and computing the coefficients of a simple Fourier series, or only solving four functional equations. The method is believed to be new. The simpler formulas of the solutions of all cases of the ring-plate boundary-value problems without any free boundary are obtained. Several examples are given.

Key words: biharmonic equation, classical Fourier-series method, functional equation, ring plate, solution formulae

1. Introduction

The lateral deflection in ring-plate problems, which are of engineering importance, is described by the single equilibrium equation [1–2]:

$$\nabla^4 w = \frac{q(r,\theta)}{D} = p(r,\theta), \tag{1}$$

where ∇^2 is the Laplace operator expressed in polar coordinates (r,θ), $q(r,\theta)$ is a continuously distributed lateral load, and D is the constant flexural rigidity.

Many techniques exist for solving the ring-plate bending problem. These techniques range from fortuitous exact solutions that are obtained by separation of variables, via numerical approximations such as finite-element and finite-difference approaches, to the various approximate energy methods such as Rayleigh-Ritz and Galerkin ([1, pp. 282–312], [2, pp. 313–328]).

How to construct exact solutions of bending problems involving a ring and plate under various loading and boundary conditions has been discussed by numerous authors ([2, pp. 313–328], [3, pp. 367–394], [4, pp. 746–785], [5–9]). The finite-form exact solutions of the symmetrical bending cases are given in [5–9]. For the general case, the classical Fourier-series method allows us to find its solution [4, pp. 746–785], [5]. By the classical Fourier-series method, the general solution of (1) can be expressed as (see [4, pp. 746–785], [5])

$$\begin{aligned} w(r,\theta) = {} & w_0(r,\theta) + c_{01}r^2 + c_{02}r^2\log r + c_{03} + c_{04}\log r \\ & + (c_{11}r^3 + c_{12}r^{-1} + c_{13}r + c_{14}r\log r)\cos\theta \\ & + (d_{11}r^3 + d_{12}r^{-1} + d_{13}r + d_{14}r\log r)\sin\theta \\ & + \sum_{n=2}^{\infty}(c_{n1}r^{n+2} + c_{n2}r^n + c_{n3}r^{-n+2} + c_{n4}r^{-n})\cos n\theta \\ & + \sum_{n=2}^{\infty}(d_{n1}r^{n+2} + d_{n2}r^n + d_{n3}r^{-n+2} + d_{n4}r^{-n})\sin n\theta, \end{aligned}$$

where $w_0(r,\theta)$ is a special solution of (1). It is well known that, for a ring-plate problem, the above solution must satisfy four boundary conditions, and by these conditions, eight linear algebraic equations for the coefficients c_{nj} and d_{nj} $(j=1, 2, 3, 4)$ are obtained. However, it is extremely difficult to solve these algebraic equations and the obtained formulae of the coefficients c_{nj} and d_{nj} $(j=1, 2, 3, 4)$ are very complex. The question arises whether or not there exists a simple method for constructing the exact solution of the ring-plate problem. The purpose of this paper is to present a simple method for solving the ring-plate problem. The method is based on the general solution of (1), which is given by (see [10–11])

$$w(r,\theta)=w_0(r,\theta)+r^2[f_1(re^{i\theta})+g_1(re^{-i\theta})]+\varphi_1(re^{i\theta})+\psi_1(re^{-i\theta}), \tag{2}$$

where $w_0(r,\theta)$ is a special solution of Equation (1) and f_1, g_1, φ_1 and ψ_1 are four arbitrary functions. Let $z=re^{i\theta}$, $\bar{z}=re^{-i\theta}$, $f_2(z)=zf_1(z)$ and $g_2(z)=zg_1(z)$, so that

$$w(r,\theta)=w_0+\bar{z}f_2(z)+zg_2(\bar{z})+\varphi_1(z)+\psi_1(\bar{z}).$$

If we take $g_2(\bar{z})=\overline{f_2(z)}$ and $\psi_1(\bar{z})=\overline{\varphi_1(z)}$, the above equation becomes

$$w(r,\theta)=w_0+\bar{z}f_2(z)+z\overline{f_2(z)}+\varphi_1(z)+\overline{\varphi_1(z)},$$

which is the standard Muskhelishvili representation [12].

The method presented in this paper can be briefly described as follows. First, substituting $f_1(z)=f(\log\frac{z}{a})$, $g_1(z)=g(\log\frac{z}{a})$, $\varphi_1(z)=\varphi(\log\frac{z}{a})$ and $\psi_1(z)=\psi(\log\frac{z}{a})$ in (2) yields

$$\begin{aligned} w(r,\theta)=w_0(r,\theta)+r^2\left[f\left(\log\frac{r}{a}+i\theta\right)+g\left(\log\frac{r}{a}-i\theta\right)\right] \\ +\varphi\left(\log\frac{r}{a}+i\theta\right)+\psi\left(\log\frac{r}{a}-i\theta\right), \end{aligned} \tag{3}$$

where f, g, φ and ψ are single-valued functions of θ. Second, by (3), the solution of the ring-plate boundary-value problem is represented in terms of four arbitrary functions that reduce to a single such function upon satisfaction of three boundary conditions. Finally, in order to satisfy the fourth boundary condition, the single arbitrary function is decomposed into a Fourier series. The method is different from the power-series method described in [13]. The author has tried to construct exact solutions of ring-plate problems by the Muskhelishvili representation [12] and find a generalized formulation of the boundary conditions similar to that posed by Tseng and Stippes [13]; however, these attempts have remained unsuccessful, since there are only two arbitrary functions in the Muskhelishvili representation and the boundary conditions considered in this paper are nonvanishing.

2. At least one built-in boundary and no free boundaries

In this section, we show how to construct exact solutions of ring-plate boundary-value problems with at least one built-in boundary and no free boundaries by the method described in the article. The boundary conditions considered here are nonvanishing, which is interesting in engineering and mathematics.

In what follows $\mathfrak{Re}[F(r,\theta)]$ means the real part of the complex function $F(r,\theta)$. We shall prove a lemma first.

Lemma 1. *If $w_0(r,\theta)$ is a special real solution of Equation (1), then the following function*

$$
\begin{aligned}
w(r,\theta) = w_2(r,\theta) &+ \frac{r^2-a^2}{b^2-a^2}\left[n'\left(\log\frac{r}{b}+\mathrm{i}\theta\right)-n'\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right]\\
&+\frac{a^2(r^2-b^2)}{(b^2-a^2)^2}\left[n\left(\log\frac{r}{b}+\mathrm{i}\theta\right)+n\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right]\\
&-\frac{a^2(r^2-b^2)}{(b^2-a^2)^2}\left[n\left(\log\frac{br}{a^2}+\mathrm{i}\theta\right)+n\left(-\log\frac{br}{a^2}+\mathrm{i}\theta\right)\right]
\end{aligned}
\tag{4}
$$

is a solution of (1) satisfying the following boundary conditions

$$
w(a,\theta)=S_0(\theta) \qquad \frac{\partial w}{\partial r}(a,\theta)=\alpha_0(\theta) \qquad w(b,\theta)=\bar{S}_0(\theta), \tag{5}
$$

where $n(z)$ is an arbitrary function, and

$$
\begin{aligned}
w_2(r,\theta) &= w_1(r,\theta)+\frac{r^2-b^2}{a^2-b^2}\mathfrak{Re}\left[\int_{-\log\frac{r}{a}+\mathrm{i}\theta}^{\log\frac{r}{a}+\mathrm{i}\theta}\alpha(-\mathrm{i}v)\mathrm{d}v\right]\\
w_1(r,\theta) &= w_0(r,\theta)+\frac{r^2-b^2}{b^2-a^2}\mathfrak{Re}\left[w_0\left(a,\theta-\mathrm{i}\log\frac{r}{a}\right)-S_0\left(\theta-\mathrm{i}\log\frac{r}{a}\right)\right]\\
&\quad+\frac{r^2-a^2}{a^2-b^2}\mathfrak{Re}\left[w_0\left(b,\theta-\mathrm{i}\log\frac{r}{b}\right)-\bar{S}_0\left(\theta-\mathrm{i}\log\frac{r}{b}\right)\right]\\
\alpha(\theta) &= \frac{a}{2}\left[\alpha_0(\theta)-\frac{\partial w_1}{\partial r}(a,\theta)\right]
\end{aligned}
$$

Proof. First step. To make the solution (3) satisfy the boundary condition $w(a,\theta)=S_0(\theta)$, we must put

$$
a^2[f(\mathrm{i}\theta)+g(-\mathrm{i}\theta)]+\varphi(\mathrm{i}\theta)+\psi(-\mathrm{i}\theta)=S_0(\theta)-w_0(a,\theta)=S(\theta). \tag{6}
$$

Letting

$$
a^2 f(\mathrm{i}\theta)+\varphi(\mathrm{i}\theta)=S(\theta)+h(\mathrm{i}\theta), \tag{7}
$$

where $h(z)$ is an arbitrary function, by (6) and (7), we have

$$
a^2 g(-\mathrm{i}\theta)+\psi(-\mathrm{i}\theta)=-h(\mathrm{i}\theta). \tag{8}
$$

The substitution $\theta=-\mathrm{i}z$ transforms (7) into

$$
a^2 f(z)+\varphi(z)=S(-\mathrm{i}z)+h(z). \tag{9}
$$

Similarly, substituting $\theta=\mathrm{i}z$ in (8) yields

$$
a^2 g(z)+\psi(z)=-h(-\mathrm{i}z). \tag{10}
$$

By (9) and (10), we find

$$
\begin{cases}
\varphi(z) = -a^2 f(z)+h(z)+S(-\mathrm{i}z),\\
\psi(z) = -a^2 g(z)-h(-\mathrm{i}z).
\end{cases}
\tag{11}
$$

Substituting (11) in (3) yields

$$w(r,\theta)=\bar{w}_1(r,\theta)+\left(r^2-a^2\right)\left[f\left(\log\frac{r}{a}+\mathrm{i}\theta\right)+g\left(\log\frac{r}{a}-\mathrm{i}\theta\right)\right]$$
$$+h\left(\log\frac{r}{a}+\mathrm{i}\theta\right)-h\left(-\log\frac{r}{a}+\mathrm{i}\theta\right), \qquad (12)$$

where

$$\bar{w}_1(r,\theta)=w_0(r,\theta)-w_0\left(a,\theta-\mathrm{i}\log\frac{r}{a}\right)+S_0\left(\theta-\mathrm{i}\log\frac{r}{a}\right).$$

Second step 2. Similar to step 1, if we let the solution (12) satisfy the boundary condition $w(b,\theta)=\bar{S}_0(\theta)$, we get

$$(b^2-a^2)\left[f\left(\log\frac{b}{a}+\mathrm{i}\theta\right)+g\left(\log\frac{b}{a}-\mathrm{i}\theta\right)\right]+h\left(\log\frac{b}{a}+\mathrm{i}\theta\right)-h\left(-\log\frac{b}{a}+\mathrm{i}\theta\right)=\bar{S}(\theta), \quad (13)$$

where $\bar{S}(\theta)=\bar{S}_0(\theta)-\bar{w}_1(b,\theta)$. Letting

$$(b^2-a^2)f\left(\log\frac{b}{a}+\mathrm{i}\theta\right)+h\left(\log\frac{b}{a}+\mathrm{i}\theta\right)=\bar{S}(\theta)+k(\mathrm{i}\theta), \qquad (14)$$

where $k(z)$ is an arbitrary function, by (13) and (14), we obtain

$$(b^2-a^2)g\left(\log\frac{b}{a}-\mathrm{i}\theta\right)-h\left(-\log\frac{b}{a}+\mathrm{i}\theta\right)=-k(-\mathrm{i}\theta). \qquad (15)$$

Equations (14) and (15) imply that

$$f(z)=\frac{1}{b^2-a^2}\left\{-h(z)+k\left(z-\log\frac{b}{a}\right)+\bar{S}\left[-\mathrm{i}\left(z-\log\frac{b}{a}\right)\right]\right\},$$
$$g(z)=\frac{1}{b^2-a^2}\left[h(-z)-k(-z+\log\frac{b}{a})\right]. \qquad (16)$$

Substituting (16) in (12), we have

$$w(r,\theta)=\bar{w}_1(r,\theta)+\frac{r^2-a^2}{b^2-a^2}\bar{S}\left(\theta-\mathrm{i}\log\frac{r}{b}\right)+\frac{r^2-b^2}{a^2-b^2}\left[h\left(\log\frac{r}{a}+\mathrm{i}\theta\right)-h(-\log\frac{r}{a}+\mathrm{i}\theta)\right]$$
$$+\frac{r^2-a^2}{b^2-a^2}\left[k\left(\log\frac{r}{a}+\mathrm{i}\theta\right)-k\left(-\log\frac{r}{a}+\mathrm{i}\theta\right)\right]. \qquad (17)$$

A straightforward computation shows that

$$\bar{w}_2(r,\theta)=\bar{w}_1(r,\theta)+\frac{r^2-a^2}{b^2-a^2}\bar{S}\left(\theta-\mathrm{i}\log\frac{r}{b}\right)$$
$$=w_0(r,\theta)+\frac{r^2-b^2}{b^2-a^2}\left[w_0\left(a,\theta-\mathrm{i}\log\frac{r}{a}\right)-S_0\left(\theta-\mathrm{i}\log\frac{r}{a}\right)\right]$$
$$+\frac{r^2-a^2}{a^2-b^2}\left[w_0\left(b,\theta-\mathrm{i}\log\frac{r}{b}\right)-\bar{S}_0\left(\theta-\mathrm{i}\log\frac{r}{b}\right)\right].$$

Noting that if $w(r,\theta)=W_1(r,\theta)+\mathrm{i}W_2(r,\theta)$, where $W_1(r,\theta)$ and $W_2(r,\theta)$ are two real functions, is a solution of (1) satisfying the boundary conditions $w(a,\theta)=S_0(\theta)$ and $w(b,\theta)=\bar{S}_0(\theta)$, we have that $W_1(r,\theta)$ is also a solution of (1) satisfying the two conditions. Thus, if we denote $w_1(r,\theta)=\mathfrak{Re}[\bar{w}_2(r,\theta)]$, then the following function

$$w(r,\theta)=w_1(r,\theta)+\frac{r^2-b^2}{a^2-b^2}\left[h\left(\log\frac{r}{a}+\mathrm{i}\theta\right)-h\left(-\log\frac{r}{a}+\mathrm{i}\theta\right)\right]$$
$$+\frac{r^2-a^2}{b^2-a^2}\left[k\left(\log\frac{r}{b}+\mathrm{i}\theta\right)-k\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right], \qquad (18)$$

is a solution of (1) satisfying the boundary conditions $w(a,\theta)=S_0(\theta)$ and $w(b,\theta)=\bar{S}_0(\theta)$.

Final step. Using (18), we compute

$$\begin{aligned}\frac{\partial w}{\partial r} = \frac{\partial w_1}{\partial r} &+ \frac{2r}{a^2-b^2}\left[h\left(\log\frac{r}{a}+\mathrm{i}\theta\right) - h\left(-\log\frac{r}{a}+\mathrm{i}\theta\right)\right] \\ &+ \frac{r^2-b^2}{(a^2-b^2)r}\left[h'\left(\log\frac{r}{a}+\mathrm{i}\theta\right) + h'\left(-\log\frac{r}{a}+\mathrm{i}\theta\right)\right] \\ &+ \frac{2r}{b^2-a^2}\left[k\left(\log\frac{r}{b}+\mathrm{i}\theta\right) - k\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right] \\ &+ \frac{r^2-a^2}{(b^2-a^2)r}\left[k'\left(\log\frac{r}{b}+\mathrm{i}\theta\right) + k'\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right].\end{aligned}$$

To make the solution (18) satisfy the boundary condition $\frac{\partial w}{\partial r}|_{r=a} = \alpha_0(\theta)$, by the above equation, we must put

$$\frac{2}{a}h'(\mathrm{i}\theta) + \frac{2a}{b^2-a^2}\left[k\left(\log\frac{a}{b}+\mathrm{i}\theta\right) - k\left(-\log\frac{a}{b}+\mathrm{i}\theta\right)\right] = \alpha_0(\theta) - \frac{\partial w_1}{\partial r}(b,\theta),$$

which implies

$$h'(\mathrm{i}\theta) = \frac{a^2}{a^2-b^2}\left[k\left(\log\frac{a}{b}+\mathrm{i}\theta\right) - k\left(-\log\frac{a}{b}+\mathrm{i}\theta\right)\right] + \alpha(\theta), \tag{19}$$

where $\alpha(\theta) = \frac{a}{2}[\alpha_0(\theta) - \frac{\partial w_1}{\partial r}(b,\theta)]$. Substituting $\theta = -\mathrm{i}z$ in (19) yields

$$h'(z) = \frac{a^2}{a^2-b^2}\left[k\left(\log\frac{a}{b}+z\right) - k\left(-\log\frac{a}{b}+z\right)\right] + \alpha(-\mathrm{i}z). \tag{20}$$

Denoting $k(z) = n'(z)$, (20) becomes

$$h'(z) = \frac{a^2}{a^2-b^2}\left[n'\left(\log\frac{a}{b}+z\right) - n'\left(-\log\frac{a}{b}+z\right)\right] + \alpha(-\mathrm{i}z).$$

After integration we obtain

$$h(z) = \frac{a^2}{a^2-b^2}\left[n\left(\log\frac{a}{b}+z\right) - n\left(-\log\frac{a}{b}+z\right)\right] + \int^z \alpha(-\mathrm{i}v)\mathrm{d}v. \tag{21}$$

Substituting $k(z) = n'(z)$ and (21) in (18) and simplifying yields

$$\begin{aligned}w(r,\theta) = w_1(r,\theta) &+ \frac{r^2-b^2}{a^2-b^2}\int_{-\log\frac{r}{a}+\mathrm{i}\theta}^{\log\frac{r}{a}+\mathrm{i}\theta} \alpha(-\mathrm{i}v)\mathrm{d}v \\ &+ \frac{r^2-a^2}{b^2-a^2}\left[n'\left(\log\frac{r}{b}+\mathrm{i}\theta\right) - n'\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right] \\ &+ \frac{a^2(r^2-b^2)}{(b^2-a^2)^2}\left[n\left(\log\frac{r}{b}+\mathrm{i}\theta\right) + n\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right] \\ &- \frac{a^2(r^2-b^2)}{(b^2-a^2)^2}\left[n\left(\log\frac{br}{a^2}+\mathrm{i}\theta\right) + n\left(-\log n\frac{br}{a^2}+\mathrm{i}\theta\right)\right],\end{aligned}$$

which implies that the following function

$$\begin{aligned}w(r,\theta) = w_2(r,\theta) &+ \frac{r^2-a^2}{b^2-a^2}\left[n'\left(\log\frac{r}{b}+\mathrm{i}\theta\right) - n'\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right] \\ &+ \frac{a^2(r^2-b^2)}{(b^2-a^2)^2}\left[n\left(\log\frac{r}{b}+\mathrm{i}\theta\right) + n\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right] \\ &- \frac{a^2(r^2-b^2)}{(b^2-a^2)^2}\left[n\left(\log\frac{br}{a^2}+\mathrm{i}\theta\right) + n\left(-\log\frac{br}{a^2}+\mathrm{i}\theta\right)\right]\end{aligned}$$

is a solution of Equation (1) that satisfies the boundary conditions (5), where

$$w_2(r,\theta)=w_1(r,\theta)+\frac{r^2-b^2}{a^2-b^2}\mathfrak{Re}\left[\int_{-\log\frac{r}{a}+\mathrm{i}\theta}^{\log\frac{r}{a}+\mathrm{i}\theta}\alpha(-\mathrm{i}v)\mathrm{d}v\right].$$

This completes the proof of Lemma 1.

From steps 1–3 we can see that multi-variable functional equations can be solved easily. It seems not to be hard to construct the solution of the ring-plate problem by Lemma 1. Unfortunately, it is very difficult to solve the single-variable functional equation. For example, if we let (4) satisfy the boundary condition $\frac{\partial w}{\partial r}(b,\theta)=\bar{\alpha}_0(\theta)$, we have

$$\frac{a^2b^2}{(a^2-b^2)^2}\left[2n(\mathrm{i}\theta)-n\left(2\log\frac{b}{a}+\mathrm{i}\theta\right)-n\left(-2\log\frac{b}{a}+\mathrm{i}\theta\right)\right]+n''(\mathrm{i}\theta)=\bar{\alpha}(\theta), \tag{22}$$

where $\bar{\alpha}(\theta)=\frac{b}{2}[\bar{\alpha}_0(\theta)-\frac{\partial w_2}{\partial r}(b,\theta)]$. Except for some simple cases, the author did not find the solution of (22).

To construct solutions of the ring-plate problem with at least a single built-in boundary and without a free boundary, we combine Lemma 1 with a Fourier-series method. Substituting $n(z)=\sum_{n=-\infty}^{\infty}a_n\mathrm{e}^{nz}$ in (4) and simplifying yields

$$w(r,\theta)=w_2(r,\theta)+\sum_{n=-\infty}^{\infty}\varphi_n(r)a_n\mathrm{e}^{\mathrm{i}n\theta}, \tag{23}$$

where

$$\varphi_n(r)=\frac{n(r^2-a^2)}{b^2-a^2}\left[\left(\frac{r}{b}\right)^n-\left(\frac{b}{r}\right)^n\right]+\frac{a^2(r^2-b^2)}{(b^2-a^2)^2}\left[\left(\frac{r}{b}\right)^n+\left(\frac{b}{r}\right)^n-\left(\frac{br}{a^2}\right)^n-\left(\frac{a^2}{br}\right)^n\right]. \tag{24}$$

We may easily verify that

$$\begin{aligned}
\varphi_{-1}(r)&=\varphi_0(r)=\varphi_1(r)=0 \qquad \varphi_n(a)=\varphi_n(b)=\varphi_n'(a)=0,\\
\varphi_{-n}(r)&=\varphi_n(r) \quad \varphi_n'(b)=\tfrac{2}{b}\left\{n^2+\tfrac{a^2b^2}{(b^2-a^2)^2}\left[2-(\tfrac{b}{a})^{2n}-(\tfrac{a}{b})^{2n}\right]\right\},\\
\varphi_n''(b)&=\tfrac{2(a^2+3b^2)n^2}{b^2(b^2-a^2)}+\tfrac{2a^2}{(b^2-a^2)^2}\left[2+(2n-1)(\tfrac{a}{b})^{2n}-(2n+1)(\tfrac{b}{a})^{2n}\right].
\end{aligned} \tag{25}$$

By use of (25), (23) can be written as

$$\begin{aligned}
w(r,\theta)&=w_2(r,\theta)+\sum_{n=2}^{\infty}\varphi_n(r)(a_n\mathrm{e}^{\mathrm{i}n\theta}+a_{-n}\mathrm{e}^{-\mathrm{i}n\theta})\\
&=w_2(r,\theta)+\sum_{n=2}^{\infty}\varphi_n(r)(A_n\cos n\theta+B_n\sin n\theta),
\end{aligned} \tag{26}$$

where $A_n=a_n+a_{-n}$ and $B_n=\mathrm{i}(a_n-a_{-n})$.

Obviously, (26) can not be used to construct the solution of the ring-plate problem with a built-in boundary and no free boundary. To do this, we must add two terms in (26). From steps 1–3, we can see that the latter part in (4), *viz.*

$$\begin{aligned}
w^h(r,\theta)=&\frac{r^2-a^2}{b^2-a^2}\left[n'\left(\log\frac{r}{b}+\mathrm{i}\theta\right)-n'\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right]\\
&+\frac{a^2(r^2-b^2)}{(b^2-a^2)^2}\left[n\left(\log\frac{r}{b}+\mathrm{i}\theta\right)+n\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right]\\
&-\frac{a^2(r^2-b^2)}{(b^2-a^2)^2}\left[n\left(\log\frac{br}{a^2}+\mathrm{i}\theta\right)+n\left(-\log\frac{br}{a^2}+\mathrm{i}\theta\right)\right],
\end{aligned} \tag{27}$$

is a solution of the homogeneous equation $\nabla^4 w = 0$ of Equation (1) satisfying the boundary-value conditions $w^h(a,\theta)=0$, $\frac{\partial w^h}{\partial r}(a,\theta)=0$ and $w^h(b,\theta)=0$.

Substituting $n(z)=\frac{1}{4}(b^2-a^2)z^2$ in (27) yields

$$w_0^h(r,\theta)=\bar{\varphi}_0(r)=(b^2-a^2)(r^2-a^2)\log\frac{r}{b}+2a^2(r^2-b^2)\log\frac{a}{b}\log\frac{r}{a}$$

which implies that

$$w_{00}(r,\theta)=A_0\bar{\varphi}_0(r) \tag{28}$$

is a solution of the homogeneous equation $\nabla^4 w=0$ of Equation (1) satisfying the boundary conditions $w_{00}(a,\theta)=0$, $\frac{\partial w_{00}}{\partial r}(a,\theta)=0$ and $w_{00}(b,\theta)=0$, where A_0 is an arbitrary constant. It is not hard to show the following:

$$\begin{aligned}
&\bar{\varphi}_0(a)=\bar{\varphi}_0'(a)=\bar{\varphi}_0(b)=0,\\
&\bar{\varphi}_0'(b)=\frac{1}{b}\left[(b^2-a^2)^2-4a^2b^2\log^2\frac{b}{a}\right],\\
&\bar{\varphi}_0''(b)=\frac{1}{b^2}\left[3b^4+2a^2b^2-a^4-4a^2b^2\left(1+\log\frac{b}{a}\right)^2\right].
\end{aligned} \tag{29}$$

Similarly, substituting $n(z)=-b(b^2-a^2)^2 z\mathrm{e}^z$ in (27), we obtain

$$w_1^h(r,\theta)=\bar{\varphi}_1(r)\mathrm{e}^{\mathrm{i}\theta} \tag{30}$$

and substituting $n(z)=-b(b^2-a^2)^2\bar{z}\mathrm{e}^{\bar{z}}$ in (27), we have

$$w_2^h(r,\theta)=\bar{\varphi}_1(r)\mathrm{e}^{-\mathrm{i}\theta}, \tag{31}$$

where

$$\bar{\varphi}_1(r)=\frac{1}{r}\left(a^2-b^2+2b^2\log\frac{b}{a}\right)(r^2-a^2)^2-\frac{1}{r}(b^2-a^2)^2\left(a^2-r^2+2r^2\log\frac{r}{a}\right).$$

Equations (30) and (31) imply that

$$w_{01}(r,\theta)=\bar{\varphi}_1(r)(A_1\cos\theta+B_1\sin\theta) \tag{32}$$

is a solution of the homogeneous equation $\nabla^4 w=0$ of Equation (1) satisfying the boundary conditions $w_{01}(a,\theta)=0$, $\frac{\partial w_{01}}{\partial r}(a,\theta)=0$ and $w_{01}(b,\theta)=0$, where A_1 and B_1 are two arbitrary constants. It is not hard to show that

$$\begin{aligned}
&\bar{\varphi}_1(a)=\bar{\varphi}_1'(a)=\bar{\varphi}_1(b)=0,\\
&\bar{\varphi}_1'(b)=4\left[(b^4-a^4)\log\frac{b}{a}-(b^2-a^2)^2\right],\\
&\bar{\varphi}_1''(b)=\frac{4}{b}\left[(3b^4+a^4)\log\frac{b}{a}-2b^2(b^2-a^2)\right].
\end{aligned} \tag{33}$$

By the principle of superposition, Equations (26), (28) and (32) imply that

$$\begin{aligned}
w(r,\theta)=\;&w_2(r,\theta)+A_0\bar{\varphi}_0(r)+\bar{\varphi}_1(r)(A_1\cos\theta+B_1\sin\theta)\\
&+\sum_{n=2}^{\infty}\varphi_n(r)(A_n\cos n\theta+B_n\sin n\theta)
\end{aligned} \tag{34}$$

is a solution of (1) satisfying the boundary conditions (5).

Next, we use (34) to construct the solutions of ring-plate problems with built-in boundaries and no free boundary. The solutions are given in the following two Theorems.

Theorem 1. *(Two built-in boundaries) The solution of the following ring-plate problem*

$$\nabla^4 w = p(r,\theta) \qquad w(a,\theta) = S_0(\theta) \qquad \frac{\partial w}{\partial r}(a,\theta) = \alpha_0(\theta),$$
$$w(b,\theta) = \bar{S}_0(\theta) \qquad \frac{\partial w}{\partial r}(b,\theta) = \bar{\alpha}_0(\theta) \tag{35}$$

is given by

$$w(r,\theta) = w_2(r,\theta) + A_0\bar{\varphi}_0(r) + \bar{\varphi}_1(r)(A_1\cos\theta + B_1\sin\theta) + \sum_{n=2}^{\infty}\varphi_n(r)(A_n\cos n\theta + B_n\sin n\theta),$$

where

$$A_0 = \frac{1}{2\pi\bar{\varphi}_0'(b)}\int_0^{2\pi}\bar{\alpha}(\theta)\mathrm{d}\theta, \qquad A_1 = \frac{1}{\pi\bar{\varphi}_1'(b)}\int_0^{2\pi}\bar{\alpha}(\theta)\cos\theta\mathrm{d}\theta,$$
$$B_1 = \frac{1}{\pi\bar{\varphi}_1'(b)}\int_0^{2\pi}\bar{\alpha}(\theta)\sin\theta\mathrm{d}\theta, \qquad A_n = \frac{1}{\pi\varphi_n'(b)}\int_0^{2\pi}\bar{\alpha}(\theta)\cos n\theta\mathrm{d}\theta,$$
$$B_n = \frac{1}{\pi\varphi_n'(b)}\int_0^{2\pi}\bar{\alpha}(\theta)\sin n\theta\mathrm{d}\theta, \qquad \bar{\alpha}(\theta) = \bar{\alpha}_0(\theta) - \frac{\partial w_2}{\partial r}(b,\theta).$$

By (34) and the boundary condition $\frac{\partial w}{\partial r}(b,\theta) = \bar{\alpha}_0(\theta)$, we have

$$A_0\bar{\varphi}_0'(b) + \bar{\varphi}_1'(b)(A_1\cos\theta + B_1\sin\theta) + \sum_{n=2}^{\infty}\varphi_n'(b)(A_n\cos n\theta + B_n\sin n\theta) = \bar{\alpha}(\theta),$$

which implies the Theorem.

Similarly, we have the following Theorem.

Theorem 2. *(The boundary $r=a$ is built-in, the boundary $r=b$ is simply supported). The solution of the following ring-plate problem*

$$\nabla^4 w = p(r,\theta), \qquad w(a,\theta) = S_0(\theta), \qquad \frac{\partial w}{\partial r}(a,\theta) = \alpha_0(\theta),$$
$$w(b,\theta) = \bar{S}_0(\theta), \qquad \left.\left[\frac{\partial^2 w}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]\right|_{r=b} = M_0(\theta) \tag{36}$$

is given by

$$w(r,\theta) = w_2(r,\theta) + A_0\bar{\varphi}_0(r) + \bar{\varphi}_1(r)(A_1\cos\theta + B_1\sin\theta) + \sum_{n=2}^{\infty}\varphi_n(r)(A_n\cos n\theta + B_n\sin n\theta),$$

where

$$A_0 = \frac{b}{2\pi[b\bar{\varphi}_0''(b) + \mu\bar{\varphi}_0'(b)]}\int_0^{2\pi}M(\theta)\mathrm{d}\theta, \quad A_1 = \frac{b}{\pi[b\bar{\varphi}_1''(b) + \mu\bar{\varphi}_1'(b)]}\int_0^{2\pi}M(\theta)\cos\theta\mathrm{d}\theta,$$

$$B_1 = \frac{b}{\pi[b\bar{\varphi}_1''(b) + \mu\bar{\varphi}_1'(b)]}\int_0^{2\pi}M(\theta)\sin\theta\mathrm{d}\theta, \; A_n = \frac{b}{\pi[b\varphi_n''(b) + \mu\varphi_n'(b)]}\int_0^{2\pi}M(\theta)\cos n\theta\mathrm{d}\theta,$$
$$B_n = \frac{b}{\pi[b\varphi_n''(b) + \mu\varphi_n'(b)]}\int_0^{2\pi}M(\theta)\sin n\theta\mathrm{d}\theta, \quad M(\theta) = M_0(\theta) - \left.\left[\frac{\partial^2 w_2}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial w_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_2}{\partial\theta^2}\right)\right]\right|_{r=b}.$$

Next, three examples are given.

2.1. EXAMPLE 1

Consider the bending of a uniformly loaded ring-plate built-in around the curve edge $r = a$, and bending moments $m(\neq 0)$ acting around the curve edge $r = b$. In this case, we have $q(r,\theta) = q_0$, $S_0(\theta) = \bar{S}_0(\theta) = \alpha_0(\theta) = 0$, and $M_0(\theta) = m$.

It is not hard to verify that $w_0(r,\theta) = Ar^4$(where $A = \frac{q_0}{64D}$) is a special solution of Equation (1). By Lemma 1, we have

$$\begin{aligned} w_1(r,\theta) &= w_0(r,\theta) + \frac{r^2-b^2}{b^2-a^2}\mathfrak{Re}\left[w_0\left(a, \theta - \mathrm{i}\log\frac{r}{a}\right)\right] + \frac{r^2-a^2}{a^2-b^2}\mathfrak{Re}\left[w_0\left(b, \theta - \mathrm{i}\log\frac{r}{b}\right)\right] \\ &= A(r^2-a^2)(r^2-b^2). \end{aligned}$$

Noting $\alpha(\theta) = \frac{a}{2}\left[\alpha_0(\theta) - \frac{\partial w_1}{\partial r}(a,\theta)\right] = Aa^2(b^2-a^2)$, again by Lemma 1, we get

$$\begin{aligned} w_2(r,\theta) &= w_1(r,\theta) + \frac{r^2-b^2}{a^2-b^2}\mathfrak{Re}\left[\int_{-\log\frac{r}{a}+\mathrm{i}\theta}^{\log\frac{r}{a}+\mathrm{i}\theta} Aa^2(b^2-a^2)\mathrm{d}v\right] \\ &= A(r^2-b^2)\left(r^2-a^2-2a^2\log\frac{r}{a}\right). \end{aligned}$$

A straightforward computation shows that

$$\begin{aligned} M(\theta) &= M_0(\theta) - \left[\frac{\partial^2 w_2}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial w_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_2}{\partial \theta^2}\right)\right]\Bigg|_{r=b} \\ &= m - 2A\left[(1+\mu)\left(b^2-a^2-2a^2\log\frac{b}{a}\right) + 4\left(b^2-a^2\right)\right]. \end{aligned}$$

Using Theorem 2, we have

$$\begin{aligned} A_0 &= \frac{m - 2Ab\left[(1+\mu)\left(b^2-a^2-2a^2\log\frac{b}{a}\right) + 4\left(b^2-a^2\right)\right]}{b\bar{\varphi}_0''(b) + \mu\bar{\varphi}_0'(b)}, \\ A_n &= B_n = 0 (n \geq 1). \end{aligned}$$

Theorem 2 implies that the solution of the boundary-value problem is given by

$$w(r,\theta) = A(r^2-b^2)\left(r^2-a^2-2a^2\log\frac{r}{a}\right) + A_0\bar{\varphi}_0(r).$$

A straightforward computation shows that the bending moment acting around the curve edge $r = a$ is $4\left[A(a^2-b^2) + A_0\left(b^2-a^2+b^2\log\frac{a}{b}\right)\right]$. The greatest deflection is attained on the circumference $r = r_0 (b \leq r_0 \leq a)$, where r_0 is a solution of the following hyperequation

$$\frac{\partial w}{\partial r} = A\left[2r\left(2r^2-a^2-b^2-2a^2\log\frac{r}{a}\right) - \frac{a^2(r^2-b^2)}{r}\right] + A_0\bar{\varphi}_0'(r) = 0,$$

which can not be solved algebraically.

It is not hard to verify that, if the inner radius b reduces to zero, the solution becomes

$$w(r,\theta) = Ar^2\left(r^2-a^2-2a^2\log\frac{r}{a}\right),$$

which is different from the solution $w(r,\theta) = A(r^2-a^2)^2$ of the circular-plate problem of uniformly loaded plate, built-in around the curve edge $r = a$ (see [10]).

2.2. Example 2

Consider the bending of a ring-shaped plate of linear varying loading, built-in around the curved edge $r=a$, and bending moments $m(\neq 0)$ acting around the curved edge $r=b$.

In this case, we have $q(r,\theta)=\frac{q_1}{a}r\cos\theta$, $S_0(\theta)=\bar{S}_0(\theta)=\alpha_0(\theta)=0$, and $M_0(\theta)=m$. It is not hard to show that the function $w_0(r,\theta)=Br^5\cos\theta$ — where $B=\frac{q_1}{192aD}$ — is a special solution of (1).

Similar to the Example 1, we have

$$w_1(r,\theta)=Br^5\cos\theta+\frac{r^2-b^2}{b^2-a^2}\mathfrak{Re}\left[Ba^5\cos\left(\theta-\mathrm{i}\log\frac{r}{a}\right)\right]$$
$$+\frac{r^2-a^2}{a^2-b^2}\mathfrak{Re}\left[Bb^5\cos\left(\theta-\mathrm{i}\log\frac{r}{b}\right)\right].$$

Noting $\cos(\theta-\mathrm{i}\log\frac{r}{a})=\frac{1}{2}(\mathrm{e}^{\log\frac{r}{a}+\mathrm{i}\theta}+e^{-\log\frac{r}{a}-\mathrm{i}\theta})=\frac{1}{2}(\frac{r}{a}+\frac{a}{r})\cos\theta+\frac{\mathrm{i}}{2}(\frac{r}{a}-\frac{a}{r})\sin\theta$, we have

$$w_1(r,\theta)=\frac{B}{2r}(r^2-a^2)(r^2-b^2)(2r^2+a^2+b^2)\cos\theta.$$

Since $\alpha(\theta)=\frac{B}{2}a(b^2-a^2)(3a^2+b^2)\cos\theta$, we have

$$w_2(r,\theta)=w_1(r,\theta)+\frac{r^2-b^2}{a^2-b^2}\mathfrak{Re}\left[\int_{-\log\frac{r}{a}+\mathrm{i}\theta}^{\log\frac{r}{a}+\mathrm{i}\theta}\frac{B}{2}a(b^2-a^2)(3a^2+b^2)\cos(-\mathrm{i}v)\mathrm{d}v\right]$$
$$=\frac{B}{2r}(r^2-a^2)(r^2-b^2)(2r^2+a^2+b^2)\cos\theta$$
$$+\frac{r^2-b^2}{a^2-b^2}\frac{B}{2}a(b^2-a^2)(3a^2+b^2)\left(\frac{r}{a}-\frac{a}{r}\right)\cos\theta$$
$$=\frac{B}{r}(r^2-a^2)^2(r^2-b^2)\cos\theta.$$

A straightforward computation shows that

$$M(\theta)=M_0(\theta)-\left[\frac{\partial^2 w_2}{\partial r^2}+\mu\left(\frac{1}{r}\frac{\partial w_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_2}{\partial\theta^2}\right)\right]\Bigg|_{r=b}$$
$$=m-\frac{2B(b^2-a^2)}{b}[(7+\mu)b^2+(1-\mu)a^2]\cos\theta$$

Using Theorem 2, we have $B_1=0$, $A_n=B_n=0 (n\geq 2)$, and

$$A_0=\frac{mb}{b\bar{\varphi}_1''(b)+\mu\bar{\varphi}_1'(b)}$$

and

$$A_1=-\frac{2B(b^2-a^2)[(7+\mu)b^2+(1-\mu)a^2]}{b\bar{\varphi}_1''(b)+\mu\bar{\varphi}_1'(b)}.$$

Theorem 2 implies that the solution of the boundary-value problem is given by

$$w(r,\theta)=A_0\bar{\varphi}_0(r)+\left[\frac{B}{r}(r^2-a^2)^2(r^2-b^2)+A_1\bar{\varphi}_1(r)\right]\cos\theta.$$

By the above solution, the greatest value of the deflection always occurs on the line $x=0(\theta=0$ or $\theta=\pi)$, and a straightforward computation shows that the bending moments acting around the curve edge $r=a$ is

$$\left[\frac{\partial^2 w_2}{\partial r^2}+\mu\left(\frac{1}{r}\frac{\partial w_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_2}{\partial\theta^2}\right)\right]\Bigg|_{r=a}=4A_0\left[b^2-a^2+(a^2+b^2)\log\frac{a}{b}\right]$$
$$+4\left[2Ba(a^2-b^2)+\frac{A_1}{a}\left(a^4-b^4+4a^2b^2\log\frac{b}{a}\right)\right]\cos\theta,$$

which implies that the largest bending moment around the curve edge $r=a$ is $4|A_0[b^2-a^2+(a^2+b^2)\log\frac{a}{b}]|+4|[2Ba(a^2-b^2)+\frac{A_1}{a}(a^4-b^4+4a^2b^2\log\frac{b}{a}]|$, and occurs at the point $(a,0)$ or (a,π).

It is clear that, if the inner radius b reduces to zero, the solution becomes

$$w(r,\theta)=Br(r^2-a^2)^2\cos\theta,$$

which is the same as the solution of the circular-plate problem with linear varying loading, built-in around the curve edge $r=a$ (see [10]).

2.3. Example 3

Consider the bending of a ring-plate with the load $\sin\frac{1}{2}\theta$, built-in around two curved edges. In this case, we have $q(r,\theta)=\sin\frac{1}{2}\theta$, $S_0(\theta)=\bar{S}_0(\theta)=\alpha_0(\theta)=\bar{\alpha}_0(\theta)=0$, and it is not hard to see that the function $w_0(r,\theta)=p_0r^4\sin\frac{1}{2}\theta$ — where $p_0=\frac{16}{945D}$ — is a special solution of (1). By Lemma 1, we have

$$w_1(r,\theta)=p_0r^4\sin\frac{1}{2}\theta+\frac{r^2-b^2}{b^2-a^2}\mathfrak{Re}\left[p_0a^4\sin\frac{1}{2}\left(\theta-\mathrm{i}\log\frac{r}{a}\right)\right]$$
$$+\frac{r^2-a^2}{a^2-b^2}\mathfrak{Re}\left[p_0b^4\sin\frac{1}{2}\left(\theta-i\log\frac{r}{b}\right)\right]=p_0\varphi(r)\sin\frac{1}{2}\theta,$$

where

$$\varphi(r)=r^4+\frac{a^4(r^2-b^2)}{2(b^2-a^2)}\left(\sqrt{\frac{r}{a}}+\sqrt{\frac{a}{r}}\right)+\frac{b^4(r^2-a^2)}{2(a^2-b^2)}\left(\sqrt{\frac{r}{b}}+\sqrt{\frac{b}{r}}\right).$$

Since $\alpha(\theta)=-\frac{p_0a}{2}\varphi'(a)\sin\frac{1}{2}\theta$, we get

$$w_2(r,\theta)=w_1(r,\theta)-\frac{a\varphi'(a)(r^2-b^2)}{2(a^2-b^2)}\mathfrak{Re}\left[\int_{-\log\frac{r}{a}+\mathrm{i}\theta}^{\log\frac{r}{a}+\mathrm{i}\theta}\sin\left(-\frac{\mathrm{i}}{2}v\right)\mathrm{d}v\right]$$
$$=\psi(r)\sin\frac{1}{2}\theta,$$

where $\psi(r)=p_0\left[\varphi(r)+\frac{a\varphi'(a)(r^2-b^2)}{a^2-b^2}\left(\sqrt{\frac{r}{a}}-\sqrt{\frac{a}{r}}\right)\right]$. By the above equation, we have

$$\bar{\alpha}(\theta)=-\frac{p_0b}{2}\left[\varphi'(b)-\frac{2\sqrt{ab}}{a+b}\varphi'(a)\right]\sin\frac{1}{2}\theta=K_0\sin\frac{1}{2}\theta.$$

Using Theorem 1, the solution of the problem can be expressed as

$$w(r,\theta)=w_2(r,\theta)+\frac{4K_0}{\pi}\left[\frac{\bar{\varphi}_0(r)}{2\bar{\varphi}_0'(b)}-\frac{\bar{\varphi}_1(r)\cos\theta}{3\bar{\varphi}_1'(b)}+\sum_{n=2}^{\infty}\frac{\varphi_n(r)\cos n\theta}{(1-4n^2)\varphi_n'(b)}\right].$$

Next, we use Lemma 1 directly to construct the solution of this boundary-value problem. In the example, the functional Equation (22) becomes

$$n''(\mathrm{i}\theta)+\frac{a^2b^2}{(a^2-b^2)^2}\left[2n(\mathrm{i}\theta)-n\left(2\log\frac{b}{a}+\mathrm{i}\theta\right)-n\left(-2\log\frac{b}{a}+\mathrm{i}\theta\right)\right]$$
$$=-\frac{b}{2}\left[\varphi'(b)+\frac{2\sqrt{ab}}{a+b}\varphi'(a)\right]\sin\frac{1}{2}\theta. \tag{37}$$

Substituting $n(z)=\delta_0\sin\frac{1}{2\mathrm{i}}z$ in (37) and simplifying yields

$$\delta_0\left\{\frac{1}{4}+\frac{a^2b^2}{(a^2-b^2)^2}\left[2-\left(\frac{b}{a}+\frac{b}{a}\right)\right]\right\}\sin\frac{1}{2}\theta=-\frac{b}{2}\left[\varphi'(b)+\frac{2\sqrt{ab}}{a+b}\varphi'(a)\right]\sin\frac{1}{2}\theta,$$

which implies $\delta_0=-\frac{2b(a+b)^2}{(a-b)^2}\left[\varphi'(b)+\frac{2\sqrt{ab}}{a+b}\varphi'(a)\right]$. Substituting $n(z)=\delta_0\sin\frac{1}{2\mathrm{i}}z$ in (4) and simplifying yields

$$w(r,\theta)=w_2(r,\theta)+\frac{\delta_0(r^2-a^2)}{2(b^2-a^2)}\left(\sqrt{\frac{r}{b}}-\sqrt{\frac{b}{r}}\right)\sin\frac{1}{2}\theta$$
$$+\frac{\delta_0a^2(r^2-b^2)}{(a^2-b^2)^2}\left(\sqrt{\frac{r}{b}}+\sqrt{\frac{b}{r}}-\sqrt{\frac{br}{a^2}}-\sqrt{\frac{a^2}{br}}\right)\sin\frac{1}{2}\theta.$$

Indeed, a straightforward computation shows that the above function is the solution of the ring-plate problem. The solution is simpler than that of the Fourier-series form.

3. Two boundaries are simply supported

In this section we shall derive the formula of the solution of the ring-plate boundary-value problem with two simply supported boundaries. Similar to Section 2, we need the following Lemma.

Lemma 2. *If $w_0(r,\theta)$ is a special real solution of Equation (1), then the following function*

$$w(r,\theta)=w_2(r,\theta)+\frac{r^2-a^2}{b^2-a^2}\left[n'\left(\log\frac{r}{b}+\mathrm{i}\theta\right)-n'\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right]$$
$$+\frac{2a^2(r^2-b^2)}{\Delta(a^2-b^2)}\left[n'\left(\log\frac{r}{b}+\mathrm{i}\theta\right)-n'\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right]$$
$$+\frac{2a^2(r^2-b^2)}{\Delta(a^2-b^2)}\left[n'\left(\log\frac{br}{a^2}+\mathrm{i}\theta\right)-n'\left(-\log\frac{br}{a^2}+\mathrm{i}\theta\right)\right]$$
$$+\frac{(1+\mu)a^2(r^2-b^2)}{\Delta(a^2-b^2)}\left[n\left(\log\frac{r}{b}+\mathrm{i}\theta\right)+n\left(-\log\frac{r}{b}+\mathrm{i}\theta\right)\right]$$
$$-\frac{(1+\mu)a^2(r^2-b^2)}{\Delta(a^2-b^2)}\left[n\left(\log\frac{br}{a^2}+\mathrm{i}\theta\right)+n\left(-\log\frac{br}{a^2}+\mathrm{i}\theta\right)\right] \tag{38}$$

is a solution of (1) satisfying the following boundary conditions

$$w(a,\theta)=S_0(\theta)\quad w(b,\theta)=\bar{S}_0(\theta),$$
$$\left[\frac{\partial^2w}{\partial r^2}+\mu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2w}{\partial\theta^2}\right)\right]\Bigg|_{r=a}=M_0(\theta), \tag{39}$$

where $n(z)$ is an arbitrary function, and

$$w_2(r,\theta)=w_1(r,\theta)+\frac{a^2(r^2-b^2)}{2\Delta}\mathfrak{Re}\left[\int_{-\log\frac{r}{a}+\mathrm{i}\theta}^{\log\frac{r}{a}+\mathrm{i}\theta}M(-\mathrm{i}v)\mathrm{d}v\right]$$
$$M(\theta)=M_0(\theta)-\left[\frac{\partial^2 w_1}{\partial r^2}+\mu\left(\frac{1}{r}\frac{\partial w_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_1}{\partial\theta^2}\right)\right]\Big|_{r=a}$$
$$\Delta=(1+\mu)a^2+(1-\mu)b^2$$

and $w_1(r,\theta)$ is given in Lemma 1.

Using (18) and the boundary condition $\left[\frac{\partial^2 w}{\partial r^2}+\mu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]\Big|_{r=a}=M_0(\theta)$, a straightforward computation shows that

$$h'(\mathrm{i}\theta)=\frac{(1+\mu)a^2}{\Delta}\left[k\left(\log\frac{a}{b}+\mathrm{i}\theta\right)-k\left(-\log\frac{a}{b}+\mathrm{i}\theta\right)\right]$$
$$+\frac{2a^2}{\Delta}\left[k'\left(\log\frac{a}{b}+\mathrm{i}\theta\right)+k'\left(-\log\frac{a}{b}+\mathrm{i}\theta\right)\right]+\frac{a^2(a^2-b^2)}{2\Delta}M(\theta).$$

Inserting $k(z)=n'(z)$ and $\theta=-\mathrm{i}z$ into the above equation yields

$$h'(z)=\frac{(1+\mu)a^2}{\Delta}\left[n'\left(z+\log\frac{a}{b}\right)-n'\left(z-\log\frac{a}{b}\right)\right]$$
$$+\frac{2a^2}{\Delta}\left[n''\left(z+\log\frac{a}{b}\right)+n''\left(z-\log\frac{a}{b}\right)\right]+\frac{a^2(a^2-b^2)}{2\Delta}M(-\mathrm{i}z).$$

Integrating the above equation yields

$$h(z)=\frac{(1+\mu)a^2}{\Delta}\left[n\left(z+\log\frac{a}{b}\right)-n\left(z-\log\frac{a}{b}\right)\right]$$
$$+\frac{2a^2}{\Delta}\left[n'\left(z+\log\frac{a}{b}\right)+n'\left(z-\log\frac{a}{b}\right)\right]+\frac{a^2(a^2-b^2)}{2\Delta}\int_0^z M(-\mathrm{i}v)\mathrm{d}v.$$

The above equation and (18) imply that Lemma 2 holds.

Substituting $n(z)=\sum_{n=-\infty}^{\infty}a_n\mathrm{e}^{nz}$ in (38) and simplifying yields

$$w(r,\theta)=w_2(r,\theta)+\sum_{n=2}^{\infty}\psi_n(r)(A_n\cos n\theta+B_n\sin n\theta), \tag{40}$$

where $A_n=a_n+a_{-n}$ and $B_n=\mathrm{i}(a_n-a_{-n})$, and

$$\psi_n(r)=\frac{n(r^2-a^2)}{b^2-a^2}\left[\left(\frac{r}{b}\right)^n-\left(\frac{b}{r}\right)^n\right]$$
$$+\frac{a^2(r^2-b^2)}{\Delta(a^2-b^2)}(2n+1+\mu)\left[\left(\frac{r}{b}\right)^n-\left(\frac{a^2}{br}\right)^n\right]+\frac{a^2\left(r^2-b^2\right)}{\Delta(a^2-b^2)}(2n-1-\mu)\left[\left(\frac{br}{a^2}\right)^n-\left(\frac{b}{r}\right)^n\right].$$

It is clear that the latter part in (38)

$$\begin{aligned}w^h(r,\theta)=&\frac{r^2-a^2}{b^2-a^2}\Big[n'\Big(\log\frac{r}{b}+\mathrm{i}\theta\Big)-n'\Big(-\log\frac{r}{b}+\mathrm{i}\theta\Big)\Big]\\&+\frac{2a^2(r^2-b^2)}{\Delta(a^2-b^2)}\Big[n'\Big(\log\frac{r}{b}+\mathrm{i}\theta\Big)-n'\Big(-\log\frac{r}{b}+\mathrm{i}\theta\Big)\Big]\\&+\frac{2a^2(r^2-b^2)}{\Delta(a^2-b^2)}\Big[n'\Big(\log\frac{br}{a^2}+\mathrm{i}\theta\Big)-n'\Big(-\log\frac{br}{a^2}+\mathrm{i}\theta\Big)\Big]\\&+\frac{(1+\mu)a^2(r^2-b^2)}{\Delta(a^2-b^2)}\Big[n(\log\frac{r}{b}+\mathrm{i}\theta)+n\Big(-\log\frac{r}{b}+\mathrm{i}\theta\Big)\Big]\\&-\frac{(1+\mu)a^2(r^2-b^2)}{\Delta(a^2-b^2)}\Big[n(\log\frac{br}{a^2}+\mathrm{i}\theta)+n\Big(-\log\frac{br}{a^2}+\mathrm{i}\theta\Big)\Big]\end{aligned}\tag{41}$$

is a solution of the homogeneous equation $\nabla^4 w=0$ satisfying the following boundary conditions

$$w(a,\theta)=0\qquad\left[\frac{\partial^2 w}{\partial r^2}+\mu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]\Bigg|_{r=a}=0\qquad w(b,\theta)=0.\tag{42}$$

Similar to Section 2, substituting $n(z)=\frac{\Delta(b^2-a^2)}{4}z^2$ in (42), we get

$$w_0^h(r,\theta)=\Delta(r^2-a^2)\log\frac{r}{b}-a^2\left[4+2(1+\mu)\log\frac{a}{b}\right]\left(r^2-b^2\right)\log\frac{r}{a}=\psi_0(r),$$

which implies that $w_{00}(r,\theta)=A_0\psi_0(r)$ is a solution of the homogeneous equation $\nabla^4 w=0$ satisfying the boundary (42), where A_0 is an arbitrary constant. Again similar to Section 2, we can also find that $w_{01}(r,\theta)=\psi_1(r)(A_1\cos\theta+B_1\sin\theta)$ is a solution of the homogeneous equation $\nabla^4 w=0$ satisfying the boundary conditions (42), where A_1 and B_1 are two arbitrary constants and

$$\psi_1(r)=-\frac{(1+\mu)(r^2-a^2)(r^2-b^2)}{\Delta r}+2\left\{r\log\frac{r}{b}+\frac{(r^2-b^2)\left[(3+\mu)a^4+(1-\mu)b^2r^2\right]}{\Delta(a^2-b^2)r}\log\frac{b}{a}\right\}.$$

By the principle of superposition, the following function, viz.

$$\begin{aligned}w(r,\theta)=&w_2(r,\theta)+A_0\psi_0(r)+\psi_1(r)(A_1\cos\theta+B_1\sin\theta)\\&+\sum_{n=2}^{\infty}\varphi_n(r)(A_n\cos n\theta+B_n\sin n\theta),\end{aligned}\tag{43}$$

is a solution of (1) satisfying the boundary conditions (39).

Using (43), it is not hard to prove the following Theorem.

Theorem 3. *(Two boundaries are simply supported) The solution of the following ring-plate problem*

$$\nabla^4 w=p(r,\theta)\quad w(a,\theta)=S_0(\theta),$$

$$\left[\frac{\partial^2 w}{\partial r^2}+\mu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]\Bigg|_{r=a}=M_0(\theta),\tag{44}$$

$$w(b,\theta)=\bar{S}_0(\theta)\left[\frac{\partial^2 w}{\partial r^2}+\mu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]\Bigg|_{r=b}=\bar{M}_0(\theta)$$

is given by

$$w(r,\theta)=w_2(r,\theta)+A_0\psi_0(r)+\psi_1(r)(A_1\cos\theta+B_1\sin\theta)+\sum_{n=2}^{\infty}\psi_n(r)(A_n\cos n\theta+B_n\sin n\theta),$$

where

$$A_0=\frac{b}{2\pi\left[b\psi_0''(b)+\mu\psi_0'(b)\right]}\int_0^{2\pi}\bar{M}(\theta)\mathrm{d}\theta,\quad A_1=\frac{b}{\pi\left[b\psi_1''(b)+\mu\psi_1'(b)\right]}\int_0^{2\pi}\bar{M}(\theta)\cos\theta\mathrm{d}\theta,$$

$$B_1=\frac{b}{\pi\left[b\psi_1''(b)+\mu\psi_1'(b)\right]}\int_0^{2\pi}\bar{M}(\theta)\sin\theta\mathrm{d}\theta,\quad A_n=\frac{b}{\pi\left[b\psi_n''(b)+\mu\psi_n'(b)\right]}\int_0^{2\pi}\bar{M}(\theta)\cos n\theta\mathrm{d}\theta,$$

$$B_n=\frac{b}{\pi\left[b\psi_n''(b)+\mu\psi_n'(b)\right]}\int_0^{2\pi}\bar{M}(\theta)\sin n\theta\mathrm{d}\theta,\quad \bar{M}(\theta)=\bar{M}_0(\theta)-\left[\frac{\partial^2 w_2}{\partial r^2}+\mu\left(\frac{1}{r}\frac{\partial w_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_2}{\partial\theta^2}\right)\right]\bigg|_{r=b}.$$

3.1. EXAMPLE 4

Consider the bending of a uniformly loaded ring-plate, with bending moments m_1 acting around the curved edge $r=a$, and bending moments m_2 acting around the curved edge $r=b$. In this case, we have $q(r,\theta)=q_0$, $S_0(\theta)=\bar{S}_0(\theta)=0$, $M_0(\theta)=m_1$, $\bar{M}_0(\theta)=m_2$. Using the Example 1, we find

$$w_1(r,\theta)=A(r^2-a^2)(r^2-b^2).$$

A straightforward computation shows that

$$M(\theta)=M_0(\theta)-\left[\frac{\partial^2 w_1}{\partial r^2}+\mu\left(\frac{1}{r}\frac{\partial w_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_1}{\partial\theta^2}\right)\right]\bigg|_{r=a}=m_1-2A\left[\Delta+2(2a^2-b^2)\right].$$

Thus, we have

$$\begin{aligned}w_2(r,\theta)&=w_1(r,\theta)+\frac{a^2(r^2-b^2)}{2\Delta}\mathfrak{Re}\left[\int_{-\log\frac{r}{a}+\mathrm{i}\theta}^{\log\frac{r}{a}+\mathrm{i}\theta}M(-\mathrm{i}v)\mathrm{d}v\right]\\&=A(r^2-b^2)\left(r^2-a^2+K_1\log\frac{r}{a}\right),\end{aligned}$$

where $K_1=\frac{\{m_1-2A[\Delta+2(2a^2-b^2)]\}a^2}{\Delta}$. Noting

$$\bar{M}(\theta)=\bar{M}_0(\theta)-\left[\frac{\partial^2 w_2}{\partial r^2}+\mu\left(\frac{1}{r}\frac{\partial w_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_2}{\partial\theta^2}\right)\right]\bigg|_{r=b}=m_2-2A\delta_0,$$

where $\delta_0=(1+\mu)(b^2-a^2)+4b^2+2K_1+(1+\mu)K_1\log\frac{b}{a}$, by the Theorem 3, we get

$$A_0=\frac{b(m_2-2A\delta_0)}{b\psi_0''(b)+\mu\psi_0'(b)}\quad A_n=B_n=0(n\geq 1).$$

Thus, the solution of the boundary-value problem is given by

$$w(r,\theta)=w_2(r,\theta)+A_0\psi_0(r).$$

For the following four numerical examples (Figures 1–4), the following parameters have been used: $a=2$, $b=1$, $\mu=0{\cdot}5$, $A=0{\cdot}05$, $m_1=A\bar{m}_1$, $m_2=A\bar{m}_2$, $\bar{m}_1=\bar{m}_2=0$ in Figure 1, $\bar{m}_1=10$ and $\bar{m}_2=0$ in Figure 2, $\bar{m}_1=10$ and $\bar{m}_2=-6$ in Figure 3, $\bar{m}_1=-20$ and $\bar{m}_2=0$ in Figure 4.

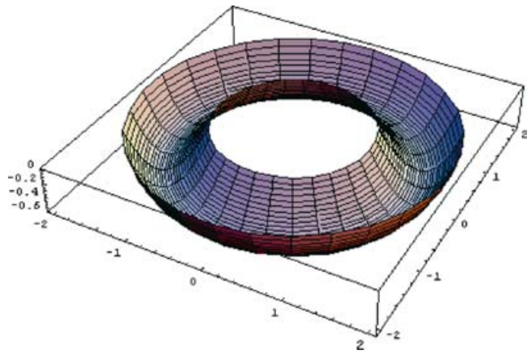

Figure 1.

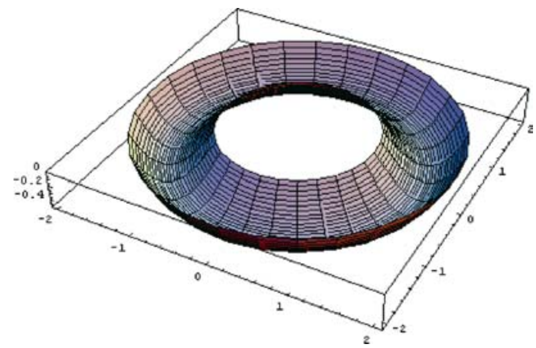

Figure 2.

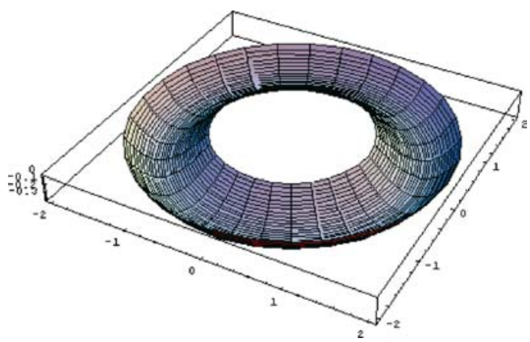

Figure 3.

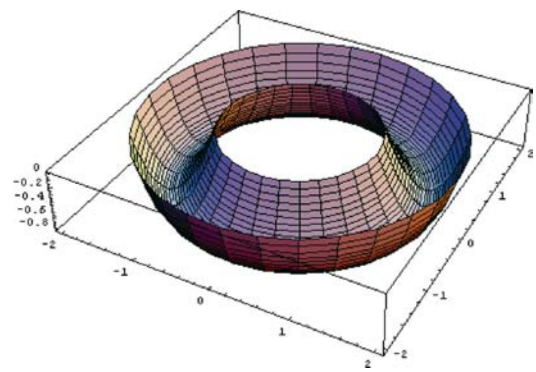

Figure 4.

3.2. Example 5

Consider the bending of a ring plate subjected to a linear varying load and simply supported around two curved boundaries. In this case, we have $q(r,\theta)=\frac{q_0}{a}r\cos\theta$ and $S_0(\theta)=\bar{S}_0(\theta)=M_0(\theta)=\bar{M}_0(\theta)=0$. Similar to the Example 4, a straightforward computation shows that the solution of the boundary-value problem is given by

$$w(r,\theta)=w_2(r,\theta)+A_1\psi_1(r)\cos\theta,$$

where

$$w_2(r,\theta)=\frac{B}{2r}(r^2-a^2)(r^2-b^2)\left(2r^2+a^2+b^2-\frac{M_0}{\Delta}\right)\cos\theta,$$

$$A_1=\frac{Bb\bar{M}_0}{b\psi_1''(b)+\mu\psi_1'(b)},$$

$$M_0=(b^2-a^2)\left[(1-\mu)b^2-(1+3\mu)a^2\right]-16a^4,$$

$$\bar{M}_0=\frac{1-\mu}{b}(b^2-a^2)\left(a^2+3b^2-\frac{M_0}{\Delta}\right)-2b\left(9b^2-a^2-\frac{2M_0}{\Delta}\right).$$

4. Conclusions

This paper presents a new method of constructing the solutions of ring-plate problems. The method can be described as follows. First, based on the general solution formula of the biharmonic equation (2), the general solution of the ring-plate problem is written in the form (3). Second, by (3) and solving three simple functional equations, the solution of the ring-plate boundary-value problem is represented in terms of a single function upon satisfaction of three boundary conditions. Third, in order to satisfy the fourth boundary condition, the single arbitrary function is decomposed into a Fourier series. Finally, by the fourth boundary condition and the Fourier-series expression of the solution, the Fourier coefficients are determined.

By this method, we have found a simpler solution formulae for ring-plate problems in the following three cases: (1) two boundaries are built-in; (2) a boundary is built-in and another is simply supported; (3) two boundaries are simply supported. Five examples have been given and four numerical solutions were provided.

There are three advantages to using the method of constructing the solution of boundary-value problems for the ring-plate geometry. First, we can construct directly the solution by the solution formulae. Second, the form of the solution obtained by the method is simpler than obtained by the classical Fourier-series method. Finally, we do not have to solve eight complex linear algebraic equations.

The method can also be used to solve boundary-value problems for the ring plate with a free boundary and some interesting mixed boundary-value problems, for example, ring plates supported at several points around two curved boundaries. The author succeeded in constructing exact solutions of the semicircular-plate problems which are built-in along the diameter edge by the method presented in the paper. The result will be given in a forthcoming paper. An interesting open problem is whether or not the solutions of all sectorial-plate problems can be constructed by the present method.

References

1. S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*. 2nd. ed. New York: McGraw-Hill Book Co. Inc. (1959)
2. P.L. Gould, *Analysis of Shells and Plates*. New York: Springer-Verlag (1988)
3. A. Clebsch, *Theorie der Elastizität Fester Körper*. Leipzig: B.G. Teubner (1862)
4. K. Beyer, *Die Statik im Stahlbeton Ban*. Berlin: Julius Springer (1948)
5. E. Reissner, Über die unsymmetrische Biegung dünner Kreisplatten. *Ingr.-Arch.* 1 (1929) 71–83.
6. A.E.H. Love, *Elastizität*. Leipzing: B.G. Teubner (1907) 562 pp.
7. A. Föppl, Die Biegung einer kreisförmigen Platten. *Sitzber. Bayer. Akad. Wiss.* 42 (1912) 164–169.
8. A.M. Wahl and G. Lobo, Stresses and deflections in flat circular plates central holds. *Trans. Amer. Soc. Mech. Eng.* 52 (1930) 29–43.
9. A. Nádai, Die Verbiegungen in einzelnen Punkten unterstützen kreisförmiger Platten. *Z. Physik* 23 (1922) 366–376.
10. P. Gao, The integral representations of solutions for circular plate problems without any hold. *Acta Mechanica* 163 (2003) 173–182.
11. J.H. Michell, The uniform torsion and flexure of incomplete tores with application to helical spring. *Proc. London Math. Soc*. 31 (1900) 130–146.
12. N.I. Muskhelishvili, *Some Basic of the Mathematical Theory of Elasticity*. New York: P. Noordhoof (1963) 518 pp.
13. C. Tseng and M. Stippes, Elastically supported circular plates. In: S. Ostrach and R.H. Scanlan (eds.), *Developments in Mechanics*. Oxford: Pergamon Press (1963) pp. 102–109.